

A Note on Supercyclic Operators

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ABSTRACT: A class $\{T_n\}$ of bounded linear operators on a Banach space X is called supercyclic if there is a vector x in X such that the scalar multiples of the orbit $\{T_n x: n \geq 1\}$ of $\{T_n\}$ is dense in X . It is proved that certain class $\{T_n\}$ of bounded linear operators on finite dimensional real Banach spaces of dimension greater than or equal to three cannot be supercyclic.

KEYWORDS: Hypercyclic operator, Supercyclic operator, Invariant subspace problem.

I. INTRODUCTION

By an operator we shall mean a linear map from a linear space into itself. Let T be an operator on a linear topological space X . The orbit of the vector $x \in X$ under the operator T is defined as $\text{Orb}(T, x) = \{T^n x: n \geq 0\}$. A vector $x \in X$ is called (a) cyclic, (b) supercyclic, (c) hypercyclic if the orbit $\text{Orb}(T, x)$ satisfies

- (a) $\overline{\text{span}(\text{Orb}(T, x))} = X$,
- (b) $\overline{C \cdot \text{Orb}(T, x)} = \{\lambda T^n x: n \geq 0, \lambda \in \mathbb{C}\} = X$,
- (c) $\overline{\text{Orb}(T, x)} = X$.

If a space X has a hypercyclic vector for an operator T then the space X has to be separable. Also, hypercyclic \Rightarrow supercyclic \Rightarrow cyclic. But the converse need not be true. A family $\{T_n\}$ of bounded linear operators on a linear topological space X is said to be hypercyclic if there exists a vector x in X such that the $\text{Orb}(T_n x: n \geq 1) = \{T_n x: n \geq 0\}$ is dense in X . The family $\{T_n\}$ is said to be supercyclic, if there is a vector x in X such that the scalar multiples the $\text{Orb}(T_n x: n \geq 1)$ is dense in X . These well known concepts originated in connection with the famous Invariant Subspace/Subset Problem (Godefroy and Shapiro[11]). So the notion of hypercyclicity is related to the famous invariant subset problem. Does every continuous linear operator T on Hilbert space H have a non-trivial closed invariant subset? Note that the scalar multiples of $\text{Orb}(T, x)$, that is, $C \cdot \text{Orb}(T, x)$ is the cone generated by $\text{Orb}(T, x)$. So the notion of supercyclicity is related to the invariant closed cone problem; that states that for a continuous linear operator T on a Hilbert space H , is it possible to find a non-trivial closed cone which is invariant under T . Since the closure of $\text{Orb}(T, x)$ is invariant for T , an operator T lacks non-trivial closed invariant subsets if and only if every non-zero vector is a hypercyclic vector for T . Similarly, an operator T lacks non-trivial closed invariant cones if and only if every non-zero vector is a supercyclic vector for T . Also an operator T lacks non-trivial closed invariant subspace if and only if every non-zero vector is a cyclic vector for T . In the case of Banach spaces there do exist operators with only trivial invariant subspaces. For, Hilbert space, the Invariant Subspace Problem (ISP) remains open.

Hilden and Wallen in 1972 have introduced the notion of supercyclicity of operators [14]. They proved that every backward unilateral weighted shift operator is supercyclic. Moreover, they showed that no normal operator on a complex Hilbert space can be supercyclic. This result is generalized for hyponormal operators by Bourdon [6]. See also [3]. Furthermore, Ansari and Bourdon [2] extended this to the class of all isometries on a Banach space. Later, supercyclicity of 2-isometric operators and more generally m -isometric operators have been investigated in [4,7,8]. In addition, supercyclicity of concave and co-concave operators have been discussed in [17]. Also, supercyclicity of convex operators have been discussed in [16]. In shortly, the study of supercyclicity of operators has received more attention in recent years. For more details, refer [3,12,4,9,10,19].

A hypercyclic vector x for T is also called a universal vector in X because every vector in X is a limit point of the orbit of x . Similarly, if x a cyclic/supercyclic vector for T every vector in X is a limit point of the $\text{span}(\text{Orb}(T, x)) / C \cdot \text{Orb}(T, x)$. Thus the study of these types of vectors seems to be of interest from the point of view of approximation theory.

Now let T be a finite dimensional real Banach space of dimension ≥ 3 , and T be a linear bounded operator on X . For $T_n = T^n, n \geq 0$, Herzog [13] proved that the family $\{T_n\}$ cannot be supercyclic. In this paper we isolate a more general family $\{T_n\}$ of linear bounded operators on finite dimensional real Banach spaces of dimension ≥ 3 which are

not supercyclic. But for some modifications the proof techniques are essentially due to Herzog [13]. The proof due to Herzog, is based on the following classical result of Rolewicz[18], for linear bounded operators acting on finite dimensional Banach spaces.

Theorem 1.1 Let T be a bounded linear operator on a finite dimensional Banach space X and let x be in X . Then the Orbit of x under T , that is $\{T^n x: n \geq 0\}$ satisfies exactly one of the following conditions

- $\lim_{n \rightarrow \infty} T^n x = 0$.
- $\lim_{n \rightarrow \infty} \|T^n x\| = +\infty$.
- The closure of the Orbit, that is $\overline{\{T^n x: n \geq 0\}}$ is compact and $0 \notin \overline{\{T^n x: n \geq 0\}}$.

II. MAIN RESULTS

Motivated by Theorem 1.1, we define the following concept class- S $\{T_n\}$ of linear operators.

Theorem 2.1 Let $\{T_n\}$ be a sequence of continuous linear operators on a Banach space X . Then $\{T_n\}$ is said to be of class- S , if it satisfies exactly one of the following for $x \in X$,

- $\lim_{n \rightarrow \infty} T_n x = 0$.
- $\lim_{n \rightarrow \infty} \|T_n x\| = +\infty$.
- The closure of the Orbit, that is $\overline{\{T_n x: n \geq 1\}}$ is compact.

Now we prove the following generalisation of Herzog's Theorem.

Theorem 2.2 Let X be a finite dimensional real Banach space of dimension ≥ 3 and let $\{T_n\}$ be a sequence of bounded linear operators on X of class- S satisfying the following conditions.

- T_n 's are invertible.
- There exists a non-trivial common invariant subspace U of dimension ≤ 2 .
- $\|T_n|_U\|$ and $\|T_n^{-1}\|$ remains bounded.

Then $\{T_n\}$ is not supercyclic.

Proof. The proof is by contradiction. Suppose $\{T_n\}$ is supercyclic on X . Then there exists $x \in X$ such that x is a supercyclic vector for $\{T_n\}$. Let W be the complement of U in X . That is, $X = U \oplus W$. W is a subspace of X . Then there exist vectors $u \in U - \{0\}$ and $w \in W - \{0\}$ such that $x = u + w$. Without loss of generality, assume that $\|T_n^{-1}\| = 1$. Consider $\{T_n w\}$. Since $\{T_n\}$ is in class- S , by the definition, we have the following three possibilities for $\{T_n w\}$.

Case 1. $\lim_{n \rightarrow \infty} T_n w = 0$.

Since $\{T_n\}$ is supercyclic there exists a sequence $\{\lambda_n\}$ of scalars and a subsequence $\{T_{n_k}\}$ of $\{T_n\}$ such that $\lambda_n T_{n_k} u + \lambda_n T_{n_k} w \rightarrow w$ as $n \rightarrow \infty$. Then $\{\lambda_n\}$ is bounded. Otherwise, we get $T_{n_k} u + T_{n_k} w \rightarrow 0$. Applying $T_{n_k}^{-1}$, we get $u = 0$, which is absurd. Let λ be a limit point $\{\lambda_n\}$. We may assume that $\lambda_n \rightarrow \lambda$ as $n \rightarrow \infty$. This implies that $T_{n_k}(\lambda u) \rightarrow w$ as $n \rightarrow \infty$. Hence $w \in U$, which is a contradiction. Therefore, $\lambda = 0$ which implies that $T_{n_k}(u) \rightarrow \infty$ as $n \rightarrow \infty$. This is a contradiction, since by assumption $\|T_n|_U\|$ is bounded. Thus case 1 is established.

Case 2. $\lim_{n \rightarrow \infty} \|T_n w\| = +\infty$.

As in Case 1, $\lambda_n T_{n_k} u + \lambda_n T_{n_k} w \rightarrow u$ as $n \rightarrow \infty$. Since $\{T_n u\}$ is bound we should have $\lambda_n \rightarrow 0$. But then we have $T_{n_k}(\lambda_n w) \rightarrow u$ as $n \rightarrow \infty$. This implies $0 = \lim_{n \rightarrow \infty} \lambda_n w = \lim_{n \rightarrow \infty} T_{n_k}^{-1} u$, which is again not possible.

Case 3. $\overline{\{T_n w: n \geq 1\}}$ is compact.

As in the previous cases, we have, $\lambda_n T_{n_k} u + \lambda_n T_{n_k} w \rightarrow u$ as $n \rightarrow \infty$. This implies $\lambda_n \rightarrow 0$, which is a contradiction. Thus, Since $\{T_n\}$ is of class- S , it follows that $\{T_n\}$ is not supercyclic. Hence the theorem.

Now we obtain Herzog's[13] Theorem as a corollary to the above theorem.

Corollary 2.3 A bounded linear operator T acting on a finite dimensional real Banach space X of dimension ≥ 3 is not supercyclic.

Proof. Let $T_n = T^n, n \geq 1$. Let if possible T is supercyclic on X . Then T is invertible, We may assume that $\|T^{-1}\| = 1$. Let $\frac{1}{\lambda}$ be an eigenvalue of T^{-1} , with $|\frac{1}{\lambda}| = 1$. Then λ is an eigenvalue of T with $|\lambda| = 1$. Let U be the eigen space

associated with λ . Then λ is an eigenvalue of T with $|\lambda| = 1$. Let U be the eigen space associated with λ . Then $\{T^n|_U\}$ is bounded and $\|T_n^{-1}\| = \|T^{-n}\| \leq 1$. So by Theorem 2.2, such a collection of operators cannot be supercyclic.

Observation 2.4 The above theorem can also be proved under the following assumptions on $\{T_n\}$. We state the modified version without proof.

Theorem 2.5 Let X be a finite dimensional real Banach space of dimension ≥ 3 and let $\{T_n\}$ be a sequence of continuous linear operators on X of class-S. Assume that there exists a subspace U of X of dimension ≤ 2 such that (1) $T_n(U) \subset U$ and (2) $\overline{\{T_n u : n \geq 1\}}$ is compact for $0 \neq u \in U$. Also assume that, $T_n(W) \subset W$ for every n , where W is the complement of U in X . Then $\{T_n\}$ is not supercyclic.

In the next theorem we isolate a class of functions of a linear operator T other than the powers of T , which are of class-S.

Theorem 2.6 Let T be a bounded linear and normal operator on a finite dimensional Hilbert space H . Let $\sigma(T)$ denotes the spectrum of T . Define a sequence of functions $\{f_n\}$ on $\sigma(T)$ such that $f_n(\lambda) \rightarrow 0$ when $|\lambda| < 1$, $|f_n(\lambda)| \rightarrow +\infty$ when $|\lambda| > 1$, $|f_n \lambda| = 1$ when $|\lambda|=1$, $\lambda \in \sigma(T)$, and f_n 's doesn't vanish on $\sigma(T)$. Then $\{f_n(T)\}$ belongs to the class-S.

Proof – Let $\dim(H) = k$ and let $\lambda_1, \lambda_2, \dots, \lambda_k$ are the distinct eigen values of T . Let $M_i = \{x \in H : (T - \lambda_i I)x = 0\}$, $1 \leq i \leq k$, and P_i be the projection of H along M_i . Then for $0 \neq x \in H$, $x = x_1 + x_2 + x_3 + \dots + x_k$, $x_i \in M_i$. Therefore, $Tx = \lambda_1 x_1 + \lambda_2 x_2 + \dots + \lambda_k x_k$. Consider the following different cases,

Case 1. $P_i x = x_i$ for some i and $P_j x = 0$ for $j \neq i$.

Now $Tx = \lambda_i x_i$. Therefore, $f_n(T)x = f_n(\lambda_i)x_i$. Then clearly $\{f_n(T)\}$ belongs to the class-S.

Case 2. $P_i x = x_i \neq 0$, for every i .

Now $f_n(T)x = f_n(\lambda_1)x_1 + f_n(\lambda_2)x_2 + \dots + f_n(\lambda_k)x_k$. In this case also it is easy to prove that $\{f_n(T)\}$ belong to class-S. Hence the theorem.

The following example gives a family of operators on a finite dimensional Banach space which does not belong to class-S.

Example 2.7 Let X be a Banach space of dimension n , and let $e_1, e_2, e_3, \dots, e_n$ be a basis of X . Let $T: X \rightarrow X$ defined by $T e_n = e_{n-1}, n \geq 1, T e_1 = 0$ and $S: X \rightarrow X$ by $Sx = \alpha x, \alpha > 1$. Clearly $T^n x \rightarrow 0$ and $S^n x \rightarrow \infty$ as $n \rightarrow \infty$, for $x \in X$. Consider $\{T_n\}$ where $T_1 = T, T_2 = S, T_3 = T^2, T_4 = S^2, T_5 = T^3, T_6 = S^3, \dots$ etc. Then $\{T_n\}$ is not in class-S.

Remark 2.8 Theorem 2.2 can be proved when X is a finite dimensional complex Banach space of dimension greater than or equal to 2.

Supercyclicity of continuous linear operators in locally convex spaces were studied in [1]. They considered doubly power bounded operators. One can think of replacing doubly power bounded operators by a family of equicontinuous functions $\{T_n\}$ acting on locally convex space with some additional conditions.

III.CONCLUSION

As already mentioned in the introduction, the study of hypercyclic/cyclic/supercyclic operators seems to be of interest in the view of approximation theory and the famous Invariant Subspace Problem. For the following applications of hypercyclic operators we refer to Kuikui Liu [15]. Applications include dynamical systems and, in particular, chaos Dynamics is the study of the evolution of the states of a system. The states are elements of some metric space X and the evolution is described by a continuous operator $T: X \rightarrow X$. In the discrete case, the progression of the states of the system are often indexed by some countable set. If $x_n \in X$ is the current system's state, then $x_{n+1} = T x_n$ gives the system's next state. Usually, when X is clear, the dynamical system (X, T) is simply written as $T: X \rightarrow X$. Another subtopic in linear dynamics is that of a chaotic system. Small changes in the system often lead to wildly different behavior. Chaotic systems are subject of great interest as there are many naturally occurring phenomena that exhibit chaotic behavior and have applications to other fields. Dynamical systems have tremendous applications as they provide tools to model various physical phenomena. They can describe the motion of various objects in Euclidean space, the flow of fluids, various engineering systems etc.

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